1. Contour integration

Basic tools. Residue theorem, etc.

2013Jan#7C. Evaluate

$$\oint_\gamma \frac{z^5}{z^7+3z-10}dz$$

where γ is parametrized by $z = 2e^{it}, 0 \le t \le 2\pi$.

2012Aug#7C. Evaluate

$$\int_0^\infty \frac{\log x}{x^2-1}dx.$$

2012Jan#9C. Evaluate

$$\lim_{x\to 0^+}\left\{\int_{-\infty}^{1-\epsilon}+\int_{1+\epsilon}^{\infty}\right\}\frac{x}{x^3-1}.$$

2011Aug#9C. Let $\log(-z)$ be a single-valued continuous branch on $\mathbb{C}\backslash[0,+\infty)$ with $\lim_{y\to 0^+}\log(-1-yi) = -\pi i$. Let $\Gamma_\epsilon^+ = \{\epsilon i + x : x \ge 0\}$ be a path of integration from $\epsilon i + \infty$ to ϵi and $\Gamma_\epsilon^- = \{-\epsilon i + x : x \ge 0\}$ be a path of integration from $-\epsilon i + \infty$ to $-\epsilon i$. Let $\Gamma_\epsilon^0 = \{\epsilon e^{i\theta : |\theta-\pi| \le \frac{\pi}{2}}\}$ be a path of integration, oriented counter clockwise. Let $\Gamma_\epsilon = \Gamma_\epsilon^+ \cup \Gamma_\epsilon^0 \cup \Gamma_\epsilon^-$.
(i) Show that

$$\phi(s) = \frac{1}{2\pi i}\int_{\Gamma_\epsilon}\frac{e^{(s-1)\log(-z)}}{e^z-1}dz$$

is independent of $\epsilon \in (0, 2\pi)$.
(ii) Show that $\phi(s)$ is an entire function in s and $\phi(n) = 0$ for $n = 2, 3, 4, \cdots$. Show that $\phi(1) \ne 0$.

2. Schwarz lemma

Basic tools. Schwarz lemma, Möbius transformations, Cauchy's integral formulas, etc.

2013Jan#8C. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function satisfying $f(1/2) = 0$ and $f(0) = 2/5$. Show that $|f'(1/2)| > 0$.
Remark: In Schwarz lemma, if f vanishes up to k-th order at 0, then one can conclude $|f(z)| \le |z|^{k+1}$.

2012Jan#7C.* Let f be a holomorphic function on $\mathbb{H} = \{z : \text{Im} z > 0\}$ satisfying $f(\mathbb{H}) \subset \mathbb{H}$ and $f(i) = i$. Show that there is a constant $C_r < \infty$ which does not depend on f such that

$$|f'(ir)| < C_r, \ 0 < r < 1.$$

3. Normal convergence

Basic tools. Morera's theorem, Montel's theorem, Hurwitz's theorem, etc.

2013Jan#9C. Let

$$F(z) = \sum_{k=1}^{\infty} (3k+5)^{-z}.$$

(i) Show that this series defines an analytic function in $\{z : \mathrm{Re} z > 1\}$.
(ii) Show that it can be continued to a meromorphic function in $\{z : \mathrm{Re} z > 0\}$. (Hint: Compare the series with $\int_0^\infty (3x+5)^{-z} dx$.)
(iii) Calculate

$$\int_\gamma F(z) dz$$

where γ is parametrized by $z = 1 + e^{it}/2, 0 \le t \le 2\pi$.
Remark: Compare this with the Riemann zeta function.

2012Aug#8C.* Let f be a holomorphic function on the unit disc $\mathbb{D}$. Fix $z_0 \in \mathbb{D}$. Suppose that $f(0) = \frac{1}{2}$, f does not vanish on $\mathbb{D}$ and $|f(z)| \le 1$. Show that $|f(z_0)| > c$ for some positive constant c independent of f.

4. Boundary values

Basic tools. Cauchy's integral formula, etc.

2012Aug#9C. Let $f(x)$ be a continuous function on $[0, 1]$. Let

$$h(z) = \frac{1}{2\pi i} \int_0^1 \frac{f(x)}{x - z} dx.$$

(a) Show that

$$\lim_{y \to 0^+} \{h(x+iy) - h(x-iy)\} = f(x), \ 0 < x < 1.$$

(b) Show that if $f \in \mathcal{C}^1((0,1))$, then $g(x) = \lim_{y\to 0^+} h(x+iy)$ exists for each $x \in (0,1)$.
Remark: These are called Sokhotski-Plemelj formulas, related to the Riemann-Hilbert problem and the Hilbert transform.

2012Jan#8C. Let $f \in L^1((0,\infty))$. Show that

$$F(z) = \int_0^\infty f(t) e^{itz} dt$$

is holomorphic on $\mathrm{Im} z > 0$. Assume that for every $x \in (-1, 1)$

$$\lim_{y\to 0^+} F(x+iy) = C,$$

where C is a constant. Show that

$$\int_0^\infty f(t)e^{itx}dt = 0, \ x \in \mathbb{R}.$$

Remark: $F(z)$ is called the Fourier-Laplace transform of f. The conclusion above implies $f = 0$.

5. Rouché's theorem

Basic tools. Rouché's theorem, etc.

2011Aug#8C. Find the number of zeros of $z^4 + z - 2$ in $\mathrm{Re}z < 0$, counting multiplicity.

6. Analytic continuation

2011Aug#7C. Let $u(z)$ be a harmonic function on

$$A : 1 < |z| < 2.$$

Show that there exists a constant α and a holomorphic function $f(z)$ on A such that

$$\mathrm{Re}f(z) = u(z) - \alpha \log |z|.$$