1. Convergence of functions

Basic tools: Lebesgue's DCT, Monotone convergence theorem, Fatou's lemma, Egorov's theorem, Lusin's theorem, convergence in measure, approximation by "good" functions, Hardy-Littlewood maximal functions, etc.

2013Jan#5. Let $f \in L^1(\mathbb{R})$. Is it true that the series

$$\sum_{n=1}^{\infty} f(x-n)$$

converges for Lebesgue almost every $x \in \mathbb{R}$?

2012Aug#4. (a) Suppose that $f_n \to f$ a.e. on $[0,1]$. Show that $f_n \to f$ on $[0,1]$ in Lebesgue measure.

(b) Suppose that $g_n \to g$ in Lebesgue measure on $[0,1]$. Show that there exists a subsequence g_{n_k} such that $g_{n_k} \to g$ a.e. on $[0,1]$.

2012Jan#5. Let $f_n \in L^p([0,1])$ with $p > 1$. Assume that $\|f_n\|_{L^p} \leq 1$ and, as $n \to \infty$, $f_n \to f$ a.e. on $[0,1]$. Show that $f \in L^p([0,1])$ and for every $1 \leq r < p$, $\|f_n - f\|_{L^r} \to 0$, as $n \to \infty$.

2011Aug#5. Assume that $\|f_n\|_{L^2([0,1])} \leq 1$ and $f_n \to f$ in Lebesgue measure as $n \to 0$. Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx$ for every $g \in L^2([0,1])$.
Added: If further assume that $\lim_{n\to\infty} \|f_n\|_{L^2([0,1])} = \|f\|_{L^2([0,1])}$, *prove that* $\|f_n - f\|_{L^2([0,1])} \to 0$, *as* $n \to \infty$.

2011Jan#4. Prove: If $E \subset \mathbb{R}$ is of Lebesgue measure zero then there exists $f \in L^1(\mathbb{R})$ so that

$$\lim_{r\to 0} \frac{1}{r} \int_{x-r}^{x+r} f(y)dy = \infty \text{ for every } x \in E. \qquad (\star)$$

Is there a set E of positive Lebesgue measure and a function $f \in L^1(\mathbb{R})$ so that $(\star)$ holds?

2011Jan#5. For $x \in \mathbb{R}$, take $K(x) = |x|^{-1/2}(1+x^2)^{-1}$ and let $K_n(x) = nK(nx), n = 1, 2, \cdots$. For $f \in L^1(\mathbb{R})$, define $T_n f(x) = \int_{\mathbb{R}} K_n(x-y)f(y)dy$. Prove or disprove that for every $f \in L^1(\mathbb{R})$, $\sup_n |T_n f(x)| < \infty$ for Lebesgue almost every x.

2010Aug#5. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions defined on $[0,1]$ is *uniformly integrable* if for every $\epsilon > 0$ there is a $\delta > 0$ so that $\sup_{n\in\mathbb{N}} |\int_E f_n dx| < \epsilon$ for all measurable subsets $E \subset [0,1]$ with measure at most δ.

Prove: If $f_n : [0,1] \to \mathbb{R}$ is a uniformly integrable sequence and $f_n(x)$ converges to $f(x)$ almost everywhere then

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

2010Jan#4. Find a sequence of bounded measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to $\frac{1}{2}\chi$, where χ is the characteristic function of the interval $[0,1]$.

Does there exist a sequence of bounded measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to 2ψ, where ψ is the characteristic function of a set of positive measure?

Recall that a sequence f_n in $L^2(\mathbb{R})$ tends weakly to $f \in L^2(\mathbb{R})$, if and only if for every $g \in L^2(\mathbb{R})$, $\int f_n g \to \int fg$.

2010Jan#6. (a) Give an example of a sequence of functions $f_n \in L^1([0,1]), n = 1, 2, \cdots$, with the following properties:

$$1. \lim_{n\to\infty} f_n(x) = 1 \text{ for any } x \in [0,1];$$

$$2. \lim_{n\to\infty} \int_0^1 |f_n(x)|dx = 2 \text{ for any } n = 1, 2, \cdots$$

(b) Show that if the f_n are as in part (a) then

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - 1|dx = 1.$$

2009Aug#4. Let $\{f_0, f_1, \cdots, f_n, \cdots\}$ be a sequence of Lebesgue measurable functions on the interval $[0,1]$.
(a) Suppose that
(i) $\lim_{n\to\infty} f_n(x) = f_0(x)$ for almost every $x \in [0,1]$;
(ii) $\int_0^1 |f_n(x)|dx < +\infty$ for every $n \geq 0$;
(iii) $\lim_{n\to\infty} \int_0^1 |f_n(x)|dx = \int_0^1 |f(x)|dx$.
Prove that $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|dx = 0$.
(b) Suppose that $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|dx = 0$.
Prove that there is a subsequence $n_k \to \infty$ such that $\lim_{k\to\infty} f_{n_k} = f_0(x)$ for almost every $x \in \mathbb{R}$.

2009Aug#5. If $f \in L^1(\mathbb{R})$ and $y > 0$, define $f_y(x) = \frac{1}{\sqrt{y}} \int_{\mathbb{R}} f(x-t)e^{-\frac{\pi t^2}{y}} dt$.
(a) Prove that for each $y > 0$, the function $f_y \in L^1(\mathbb{R})$; i.e. $\int_{\mathbb{R}} |f_y(t)|dt < +\infty$ for each $y > 0$.
(b) Prove that $\lim_{y\to 0} \int_{\mathbb{R}} |f(x) - f_y(x)|dx = 0$.
(c) There exists a constant $C > 0$ such that for every $f \in L^1(\mathbb{R})$,

$$\left|\{x \in \mathbb{R} : \sup_{y>0} |f_y(x)| > \lambda\}\right| \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f(t)|dt.$$

Using this inequality, prove that if $f \in L^1(\mathbb{R})$, then for almost every $x \in \mathbb{R}$, $\lim_{y \to 0} f_y(x) = f(x)$.