1. Theory of Distributions

Basic tools: Approximations to the identity, Fourier transform, etc.

2013Jan#8R. Let T be a distribution on $\mathbb{R}$. Set $\tau_a\phi(x) = \phi(x-a)$ and assume that $\langle T, \tau_a\phi\rangle = \langle T, \phi\rangle$ for all $a \in \mathbb{R}$ and all test functions ϕ. Prove that T is a constant.

2013Jan#9R. Let δ be the Dirac delta function concentrated at zero and δ' be its derivative in the sense of distributions. Consider

$$f_1(x) = \begin{cases} 0 & |x| > 1 \\ \sin(x^3) & |x| < 1 \end{cases}, \qquad f_2(x) = \delta'(x).$$

For $j = 1, 2$, find $\alpha_j = \sup\{\alpha : f_j \in H^\alpha(\mathbb{R})\}$, where $H^\alpha(\mathbb{R})$ is the Sobolev space of order α.

2012Aug#9R. Recall that a distribution T on $\mathbb{R}$ has order 0 if for each compact subset K of $\mathbb{R}$,

$$|\langle T, \varphi\rangle| \le C_K \max_{x\in\mathbb{R}} |\varphi(x)|, \quad \mathrm{supp}\varphi \subset \mathbb{R}, \varphi \in C^\infty(\mathbb{R}).$$

Let $0 < b_{n+1} < a_n < b_n$. Let $\chi_{[a_n,b_n]}$ be the characteristic function of $[a_n, b_n]$ and

$$f = \sum_{n=1}^{\infty} c_n \chi_{[a_n,b_n]}, \quad c_n \in \mathbb{R}.$$

Assume that $f \in L^1(\mathbb{R})$.

(a) Prove that the distribution derivative f' has order 0 if $\sum |c_n| < \infty$.

(b) Prove that f' does not have order 0 if $\sum |c_n| = \infty$.

2012Jan#9R. Consider $f(x) = |x|^{-n}$ on $\mathbb{R}^n\backslash\{0\}$. Does there exist a distribution T on $\mathbb{R}^n$ such that

$$\langle T, \phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

for all $\phi \in C^\infty(\mathbb{R}^n)$ which have compact support in $\mathbb{R}^n\backslash\{0\}$?

2011Aug#9R. Assume that f is a distribution in $\mathcal{D}'(\mathbb{R}^2)$ and $f_x = 0$ and $f_y = 0$ in the sense of distributions. Prove that f is a constant.

2011Jan#8R. Let $f \in L^\infty(\mathbb{R})$ and

$$\int_{\mathbb{R}} e^{-(x-y)^2} f(y) = 0$$

for any $x \in \mathbb{R}$. Prove that $f(x) = 0$ for Lebesgue almost every $x \in \mathbb{R}$.

2011Jan#9R. Let f be a continuous function on $\mathbb{R}^2$. Assume that for every $s \in \mathbb{R}$, as functions in $t \in \mathbb{R}$, the distributional derivatives $\frac{d}{dt} f(t,s) \equiv A_s(t)$ and $\frac{d}{dt} f(s,t) \equiv B_s(t)$ are in $L^\infty(\mathbb{R})$. (However, it is not assumed that $\partial_{x_1} f(x_1,x_2), \partial_{x_2} f(x_1,x_2)$ exist in the sense of distributions on $\mathbb{R}^2$.) Suppose that

$$\sup_{s\in\mathbb{R}} \{\|A_s\|_{L^\infty(\mathbb{R})} + \|B_s\|_{L^\infty(\mathbb{R})}\} < \infty.$$

(i) Let χ be a smooth function on $\mathbb{R}^2$ with compact support and $\int \chi(x)dx = 1$. Let

$$f_\epsilon(x) = \int f(x+\epsilon y)\chi(y)dy.$$

Show that there is a constant C such that

$$|\partial_{x_1} f_1(x_1,x_2)| \le C, \quad (x_1,x_2) \in \mathbb{R}^2, \quad \epsilon > 0.$$

(ii) Show that f is (locally) a Lipschitz function.

2010Aug#8R. In both parts either construct a distribution in $\mathcal{D}'(\mathbb{R})$ and prove the appropriate estimates, or show that no such distribution exists.

(i) Is there a distribution $u \in \mathcal{D}'(\mathbb{R})$ so that for all $\phi \in C_0^\infty(\mathbb{R})$ with compact support in $(0,\infty)$ one has

$$\langle u, \phi \rangle = \int \phi(x)|x|^{-5}dx?$$

(ii) Is there a distribution $v \in \mathcal{D}'(\mathbb{R})$ so that for all $\phi \in C_0^\infty(\mathbb{R})$ with compact support in $(0,\infty)$ one has

$$\langle v, \phi \rangle = \sum_{n=1}^{\infty} \phi^{(n)}(2^{-n})?$$

2010Jan#0R. We shall use the following normalization for the Fourier transform on $\mathbb{R}^n$. When it makes (classical) sense:

$$\hat{f}(\xi_1, \cdots, \xi_n) = \int_{\mathbb{R}^n} f(y_1, \cdots, y_n) e^{-i(y_1\xi_1 + \cdots + y_n\xi_n)} dy_1 \ldots dy_n.$$

And the Fourier inversion formula is thus, when it makes sense:

$$f(\xi_1, \cdots, \xi_n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi_1, \cdots, \xi_n) e^{i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \ldots d\xi_n.$$

1) Let f be a a bounded function on $\mathbb{R}^2$, explain how (in the theory of tempered distributions) one defines its Fourier transform $\hat{f}$, even if f is not integrable. What is the Fourier transform of the constant function 1?

2) Let φ be a continuous function with compact support on $\mathbb{R}$. Its Fourier transform is therefore a continuous function $\hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x)e^{-ix\xi}dx$. Let ϕ be the function of two variables defined by

$$\phi(x_1, x_2) = \varphi(x_1)$$

Find the Fourier transform of ϕ, in terms of $\hat{\varphi}$?

2009Aug#8R. Let g be a positive decreasing function defined on $(0, +\infty)$. Show that the following are equivalent:

(a) There exists a distribution T on $\mathbb{R}$ such that $\langle T, \varphi\rangle = \int_0^\infty \varphi(x)g(x)dx$ for all test functions $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ which have compact support in the open interval $(0, \infty)$.

(b) There exists a non-negative integer $k \in \mathbb{N}$ and a constant $C > 0$ such that for all $x \in (0, 1), g(x) \leq Cx^{-k}$.