2012Jan#3.* Let A be an $n \times n$ invertible real matrix. Show that there is a complex matrix B such that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{B^k}{k!} = A.$$

Here $B^0 = I$ is the identity $n \times n$ matrix.

2010Jan#3. For $\lambda > 0$, set

$$F(\lambda) = \int_0^1 e^{-10\lambda x^4 + \lambda x^6} dx.$$

Prove that there exist A and $C > 0$ such that $F(\lambda) = A\lambda^{-1/4} + E(\lambda)$, where $|E(\lambda)| \le C\lambda^{-1/2}$.

2009Jan#4. For $\lambda > 0$, define $H(\lambda) = \int_0^\infty e^{-\lambda(x^3+x^5)} dx$. Prove that there are positive constants A and C so that $\left|H(\lambda) - A\lambda^{-\frac{1}{3}}\right| \le C\lambda^{-1}$ for $\lambda > 1$.

Proof. The integrand is decaying exponentially if x stays away from 0; when x is close to 0, $x^3 + x^5 \approx x^3$, hence x^3 has the major effect to the integral. Observe that by change of variable

$$\int_0^\infty e^{-\lambda x^3} dx = A\lambda^{-1/3}$$

for some positive constant A. Hence the problem is to show that the "perturbation" of the above integral by the extra term x^5 is of order λ^{-1}, i.e.

$$\left|\int_0^\infty e^{-\lambda(x^3+x^5)} - e^{-\lambda x^3} dx\right| \le C\lambda^{-1}$$

for some positive constant C.

First observe that the integral on $[1, \infty)$ is decaying rapidly. Indeed,

$$\begin{aligned}
\int_1^\infty e^{-\lambda(x^3+x^5)} dx &\le \int_1^\infty e^{-\lambda x^3} dx \\
&= \int_1^\infty e^{-\lambda t} \frac{1}{3} t^{-2/3} dt \\
&\le \int_1^\infty e^{-\lambda t} dt \\
&= \frac{e^{-\lambda}}{\lambda} \le \frac{1}{\lambda}.
\end{aligned}$$

Hence it suffices to show

$$\int_0^1 e^{-\lambda x^3}(1 - e^{-\lambda x^5}) dx \le C\lambda^{-1}.$$

Second, we notice that when x is close to 0, $(1 - e^{-\lambda x^5})$ is small. More precisely, we have estimate

$$1 - e^{-\lambda x^5} \leq \lambda x^5.$$

This follows from the general fact that $1 - e^{-t} \leq t$ for all $t \geq 0$ which can be seen by differentiating both sides. Using this the problem reduces to showing

$$\int_0^1 e^{-\lambda x^3} \lambda x^5 dx \leq C\lambda^{-1}.$$

But

$$\begin{aligned}
\lambda \int_0^1 e^{-\lambda x^3} x^5 dx &\leq \lambda \int_0^\infty e^{-\lambda x^3} x^5 dx \\
&= \lambda \int_0^\infty e^{-\lambda t} t^{5/3} \frac{1}{3} t^{-2/3} dt \\
&= \frac{\lambda}{3} \int_0^\infty e^{-\lambda t} t dt \\
&= \frac{\lambda}{3} \frac{1}{\lambda^2} \int_0^\infty e^{-s} s ds \\
&= C\lambda^{-1}.
\end{aligned}$$

This finishes the proof.

□