1. Integration/summation by parts

2013Jan#1. Prove that

$$\int_0^1 \frac{dx}{x^x} = \sum_{n=1}^{\infty} \frac{1}{n^n}$$

Hint: Use the Taylor expansion for the exponential function.

2012Aug#2. Prove that

$$\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan t, \quad t > 0.$$

Remark: The left is the Laplace transform of $\mathrm{sinc}(x)$, *related to the Dirichlet integral.*

2011Jan#3.* Show that there exists a constant C such that for all $x \in [0, 2\pi]$ and $n = 1, 2, \cdots$

$$\Big| \sum_{k=1}^{n} \frac{\sin(kx)}{k} \Big| < C$$

Hint: Break the sum into two parts for $kx < 1$ *and* $kx \geq 1$, *respectively.*
Remark: This is the Fourier series of a sawtooth wave.

2009Aug#6. (a) For which real numbers $a \in \mathbb{R}$ and $b > 0$ is it true that $\big| \int_0^N e^{ix^b} (1+x)^a dx \big|$ is bounded independently of the number $N > 0$?

(b) For which real numbers $a \in \mathbb{R}$ and $b > 0$ is it true that the improper integral $\int_0^\infty e^{ix^b} (1+x)^a dx$ converges?