1. Sequences and series

Basic tools: Cauchy's tests, Taylor expansions, estimating sums by integrals, alternating series, etc.

2013Jan#3. Suppose $\{x_n\}$ is a numerical sequence such that

$$0 < x_{n+m} \leq x_n + x_m$$

for all $n, m \in \mathbb{N}$ and $0 < x_1$. Prove that

$$\lim_{n\to\infty} \frac{x_n}{n}$$

exists.
Hint: Prove that $\lim_{n\to\infty} x_n/n = \liminf_{n\to\infty} x_n/n$.
Remark: This is called Fekete's subadditive lemma.

2012Aug#1. Let a_n, b_n be real numbers.

(a) Assume that $\sum a_n$ converges. Show that there exists a sequence $b_n \to +\infty$ such that $\sum a_n b_n$ still converges.

(b) Let b_n be an unbounded sequence. Show that there is a convergent series $\sum a_n$ such that $\sum a_n b_n$ diverges.

Added: (c) Assume that $\sum a_n$ *diverges. Show that there exists a sequence* $b_n \to 0$ *such that* $\sum a_n b_n$ *still diverges.*

2011Aug#1.* Let

$$f(x, y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}, \quad y \neq 0.$$

(i) Show that for each $y > 0$, $g(y) = \lim_{x\to\infty} f(x, y)$ exists. Evaluate $g(y)$.
(ii) Determine if $f(x, y)$ converges uniformly to $g(y)$ for $y \in (0, \infty)$ as $x \to \infty$.

2010Aug#2. Let

$$s_N(x) = \sum_{n=1}^{N} (-1)^n \frac{x^{3n}}{n^{2/3}}.$$

Prove that $s_N(x)$ converges to a limit $s(x)$ on $[0, 1]$ and that there is a constant C so that

$$\sup_{x\in[0,1]} |s_N(x) - s(x)| \leq CN^{-2/3}$$

holds.

2009Aug#1. Let $b \geq 1$. A sequence $\{a_n\}_{n=0}^{\infty}$ of positive real numbers is defined inductively by specifying $a_0 > 0$, and then setting

$$a_{n+1} = \frac{a_n}{2} + \frac{b^3}{2a_n^2}$$

for $n \geq 0$.
(a) Show that if $L = \lim_{n\to\infty} a_n$ exists, then $L = b$.
(b) Show that there is an open interval $I_b \subset \mathbb{R}$ containing b so if $a_0 \in I_b$, then $L = \lim_{n\to\infty} a_n$ exists.
(c) What can you say about the length of I_b.

2009Aug#2. Let $f(x) = \sum_{n=1}^{\infty}(1 + n^4x^2)^{-1}$.
(a) Show that f is continuously differentiable on $(0, \infty)$.
(b) Show that there is a constant $C > 0$ so that $f(x) \leq Cx^{-\frac{1}{2}}$ for all $0 < x \leq 1$, and $f(x) \leq Cx^{-2}$ for $x \geq 1$.
(c) Show that the improper Riemann integral $\int_0^\infty f(x)dx = \lim_{\substack{\epsilon\to 0 \\ N\to\infty}} \int_\epsilon^\infty f(x)dx$ exists.