1. Hilbert spaces, Banach spaces

Basic tools: Uniform boundedness principle, closed graph theorem, total boundedness, Arzelà-Ascoli theorem, Banach fixed-point theorem, L^p spaces, Cantor's diagonal argument, etc.

2013Jan#7R. Let H be a Hilbert space, and let A be a linear operator defined on all of H satisfying $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$. Prove that A is a bounded operator.
Remark: This is called Hellinger-Toeplitz theorem
Added: (i) Show that if $\langle Ax, y\rangle = \langle x, Ay\rangle$ holds only for x, y in a dense subspace H_0, then the conclusion may fail. (ii) If instead $\langle Ax, y\rangle = \langle x, Ay\rangle$ holds for $x \in H_0$ and $y \in H$, then the conclusion still holds.

2012Aug#7R. Suppose X, Y, Z are Banach spaces, and $B : X \times Y \to Z$ is a map such that $B(x, y)$ is linear and continuous in x when y is fixed, and it is linear and continuous in y when x is fixed. Show that there is a constant C such that

$$\|B(x, y)\|_Z \le C\|x\|_X\|y\|_Y, \quad x \in X, y \in Y.$$

2012Aug#8R. Let $f_n \to f$ weakly in $L^2(\mathbb{R})$ and $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$. Show that $f_n \to f$ strongly in $L^2(\mathbb{R})$.

2012Jan#7R. Let x_n be a sequence in a Hilbert space H. Assume that, as $n \to \infty$, x_n converges weakly to $A \in H$. Show that there is a subsequence x_{n_k} such that the sequence

$$\frac{1}{N}\sum_{k=1}^{N} x_{n_k}$$

converges strongly to A in H.
Remark: This is called Banach-Saks theorem.

2011Aug#3. Let E be the vector space of bounded sequences of real numbers $x = (x_1, x_2, \cdots)$. Define norms

$$\|x\|_1 = \sup |x_n|, \quad \|x\|_2 = \sup \frac{|x_n|}{n}.$$

Let B be the set of E with $\|x\|_1 \le 1$.
(i) Prove or disprove that B is compact in $(E, \|\cdot\|_1)$.
(ii) Prove or disprove that B is compact in $(E, \|\cdot\|_2)$.

2011Aug#8R. (i) What are the necessary and sufficient conditions on $\lambda_n > 0$ for the set

$$\{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : |x_n| \le \lambda_n, \forall n\}$$

to be compact in $\ell^2(\mathbb{N})$?
(ii) What are the necessary and sufficient conditions on $\mu_n > 0$ for the set

$$\{(x_1, x_2, \cdots) \in \ell^2(\mathbb{N}) : \sum_n \frac{|x_n|^2}{\mu_n^2} \le 1\}$$

to be compact in $\ell^2(\mathbb{N})$?

2011Jan#1. Let K be a continuous function on the square $[0,1] \times [0,1]$ and let g be a continuous function on $[0,1]$. Show that there is a unique continuous function f on $[0,1]$ so that

$$f(x) = \int_0^x K(x,y)f(y)dy + g(x).$$

2011Jan#6.* Consider the Hilbert space $L^2([0,1])$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal system of functions in $L^2([0,1])$.
(i) Suppose that $e_n \in C^1([0,1])$ for all $n \in \mathbb{N}$ (i.e. the e_n have continuous derivatives). Prove that

$$\sup_n \max_{x\in[0,1]} |e_n'(x)| = \infty.$$

(ii) Suppose that $\{e_n\}_{n=1}^\infty$ is complete (i.e. if $g \in L^2([0,1])$ and $\langle g, e_n\rangle = 0$ for all $n \in \mathbb{N}$ then $g = 0$ almost everywhere). Prove that

$$\sum_{n=1}^\infty |e_n(x)|^2 = \infty \text{ almost everywhere.}$$

2010Aug#3. Let K be a continuous function on the unit square $Q = [0,1] \times [0,1]$ with the property that $|K(x,y)| < 1$ for all $(x,y) \in Q$. Show that there is a continuous function g defined on $[0,1]$ so that

$$g(x) + \int_0^1 K(x,y)g(y)dy = \frac{e^x}{1+x^2}, \quad 0 \le x \le 1.$$

2010Aug#9R. Assume that the sequence $\{x_n\}$ of real numbers is such that $x_n \neq 0$ for some n. Take $p \in (1,\infty)$ and let G be the set of all sequences $\{y_n\}$ so that $\{y_n\} \in \ell^p$ and

$$\lim_{N\to\infty} \sum_{n=1}^N y_n x_n = 0.$$

Prove that G is dense in ℓ^p if and only if $\{x_n\} \notin \ell^p$ where $q^{-1} + p^{-1} = 1$.

2010Jan#7R. Let W be the space of continuous functions f on $[0,1]$, whose distributional derivative on $(0,1)$, is an integrable function.

In one variable, this simply means that $f(x) = f(0) + \int_0^x g(t)dt$, for some integrable function g, and then $f' = g$. On W one considers the norm defined by

$$\|f\|_W = |f(0)| + \int_0^1 |f'(t)|dt.$$

Let Λ be the space of continuous functions on $[0, 1]$, that are Hölder continuous of order $\frac{1}{2}$ (i.e. function f such that for some constant $C > 0$, for every x and y, $|f(x) - f(y)| \le C|x - y|^{\frac{1}{2}}$). On Λ one considers the norm

$$\|f\|_\Lambda = |f(0)| + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|^{\frac{1}{2}}}.$$

Equipped with these norms W and Λ are Banach spaces. And it is immediate that the diagonal $\Delta \subset W \times \Lambda$, that is the set of all (f, f) for $f \in W \cap \Lambda$, is a closed subspace of $W \times \Lambda$. You are not asked to justify the above.
1) Show that if $f \in W$ and $f' \in L^2$ (not only $\in L^1$), then $f \in \Lambda$.
2) For each integer $N > 0$, set $u_N(x) = \frac{1}{N}\sin(Nx)$. Find constants A_N such that, $A_N \to 0$ as $N \to \infty$, and for every x and $y \in \mathbb{R}$

$$|u_N(x) - u_N(y)| \le A_N|x - y|^{\frac{1}{2}}.$$

3) Prove that there is a Hölder continuous function f, of of Hölder exponent $\frac{1}{2}$, defined on $[0, 1]$ (i.e. $f \notin \Lambda$) whose distributional derivative on $(0, 1)$, is not an integrable function (i.e. $f \notin W$).
Even if you were not able to prove the result of question 2, you can use it for applying the open mapping theorem to the projection of Δ on Λ, when arguing by contradiction.

2010Jan#8R. Let H be a Hilbert space on $\mathbb{R}$, with scalar product denoted by $\langle .,. \rangle$, and associated norm denoted by $\| \cdot \|$.
1) Assume that (x_n) and (y_n) are sequences in H such that $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\langle x_n, y_n \rangle \to 1$ as $n \to \infty$. Show that $(x_n - y_n) \to 0$ as $n \to \infty$.
2) Let T be a continuous linear map from H into itself.
2.1) Recall what is the definition of the adjoint operator T^*.
2.2) Assume that T is self adjoint, i.e. that $T^* = T$. And assume that, for some sequence $x_n \in H$, with $\|x_n\| \le 1$:

$$1 = \sup_{\|x\| \le 1} \|T(x)\| = \lim_{n \to \infty} \|T(x_n)\|.$$

Using question 1, show that $T^2(x_n) - x_n$ tends to 0 as $n \to \infty$.
Conclude that at least one of the 2 operators $T + \mathbf{1}$ or $T^* - \mathbf{1}$ is not invertible. Here $\mathbf{1}$ denotes the identity map on H (i.e. $\mathbf{1}(x) = x$).

2009Aug#3. Let X and Y be normed vector spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$. Let $\Omega \subset X$ be an open set and let $x \in \Omega$. Recall that a function $F : \Omega \to Y$ is differentiable at $\Omega \subset X$ if there is a continuous linear transformation $S_x : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - S_x(h)\|_Y}{\|h\|_X} = 0.$$

We then say that S_x is the derivative of F at x.

(a) Let X, Y, and Z be normed vector spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$. Let $F : X \to Y$ be differentiable at a point $x_0 \in X$ with derivative S_{x_0}, and let $G : Y \to Z$ be differentiable at the point $F(x_0)$ with derivative $T_{F(x_0)}$. Prove that the composition $G \circ F : X \to Z$ defined by $G \circ F(x) = G(F(x))$ is differentiable at x_0, and compute its derivative.

(b) Let M_n denote the space of all real $n \times n$ matrices m, and define $F : M_n \to M_n$ by $F(m) = m^3$. Prove that F is differentiable at every matrix $m \in M_n$, and compute the derivative of F.

(c) Let $\Omega \subset M_n$ denote the set of invertible $n \times n$ matrices, and define $G : \Omega \to M_n$ by setting $G(m) = m^{-1}$. Prove that Ω is an open subset of M_n, and that G is differentiable at every point $m \in \Omega$, and compute the derivative of G.

2009Aug#7R. (a) Let H_1 and H_2 be Hilbert spaces, and let $T : H_1 \to H_2$ be a continuous linear operator. Give a precise definition of the adjoint operator T^*.

(b) Let $(a, b) \subset \mathbb{R}$ be a (possibly infinite) open integral. If $f \in L^2(a, b)$, explain what it means that the distributional derivative f' is also in $L^2(a, b)$.

(c) Let $\mathbb{R}^+$ denote the positive real axis $[0, \infty)$. Let $H^1(\mathbb{R})$ (respectively $H^1(R^+)$) be the space of real-valued functions $f \in L^2(\mathbb{R})$ (respectively $f \in L^2(\mathbb{R}^+)$) such that the distributional derivative f' is also in $L^2(\mathbb{R})$ (respectively $L^2(\mathbb{R}^+)$). Then $H^1(\mathbb{R})$ and $H^1(\mathbb{R}^+)$ are Hilbert spaces with inner product given by

$$\langle f, g\rangle_{H^1(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)dx + \int_{\mathbb{R}} f'(x)g'(x)dx;$$

$$\langle f, g\rangle_{H^1(\mathbb{R}^+)} = \int_{\mathbb{R}^+} f(x)g(x)dx + \int_{\mathbb{R}^+} f'(x)g'(x)dx.$$

Let $T : H^1(\mathbb{R}) \to H^1(\mathbb{R}^+)$ be the mapping given by restriction. Compute explicitly the adjoint operator T^*.