1 Salem-Bluhm's construction of Salem sets

after R. Salem [1] and C. Bluhm [2]
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Abstract

Given $\alpha \in (0,1)$, we construct a random Cantor set whose Fourier and Hausdorff dimensions equal α almost surely.

1.1 Introduction

Recall that in the construction of the standard $\frac{1}{3}$ Cantor set there are three ingredients: the dissection number 2, the dissection ratio $\frac{1}{3}$ and the positions of the subintervals 0 and $\frac{2}{3}$ (which we will call translations).

Given $\alpha \in (0,1)$, we are interested in constructing a Cantor-type set whose Fourier dimension (see below for definition) and Hausdorff dimension are both equal to α. Such sets are called Salem sets and were first constructed by Salem [1]. They are special in the sense that they close the gap between the Fourier and Hausdorff dimensions (it is a general fact that the former can not exceed the latter).

Salem achieved this by randomizing the dissection ratios and picking incommensurable translations in the construction of Cantor set. On the other hand one can also instead randomize the translations in order to obtain Salem sets. Both approaches increase the dissection number at each step in order to make up for the ε-loss of decay in the case without such increments. We will follow the second approach which was introduced by Bluhm [2].

In what follows, we will restrict ourselves to $\mathbb{R}^1$. All measures are defined on Borel σ-algebra. The Fourier transform of a finite measure μ is defined by $\hat{\mu}(\xi) = \int e^{i\xi t}\mu(dt)$.

1.2 The main result

Theorem 1. *Given $\alpha \in (0,1)$, there exists a compact set $K \subset [0,1]$ and a probability measure μ supported on K, such that*
(i) K has Hausdorff dimension α
(ii) for all $\beta < \alpha$, $\hat{\mu}(\xi) = O(|\xi|^{-\beta/2})$ as $|\xi| \to \infty$
(iii) $\mu(I) \lesssim |I|^\alpha$ for all interval I.

1.3 The set

In fact we will construct a class of K most of which will have the properties stated in the theorem.

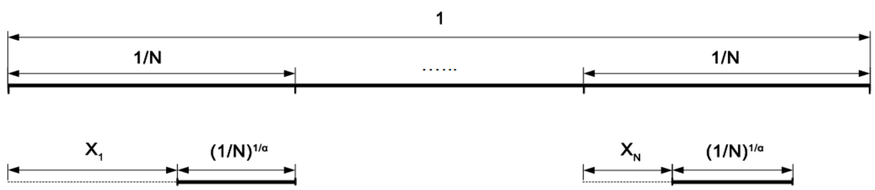
The construction will start with the nominal second step. In the N-th step of the construction, the dissection number will be precisly N, the dissection ratio will be denoted by θ_N, the translations by $X_{N,j}$, where $j = 1, \cdots, N$.

More precisely, for $N \in \mathbb{N}, N \geq 2$, let $\theta_N = N^{-\frac{1}{\alpha}}$. Notice that

$$N^{-1} - \theta_N = N^{-1}(1 - N^{-(\frac{1}{\alpha}-1)}) > c_\alpha N^{-1}.$$

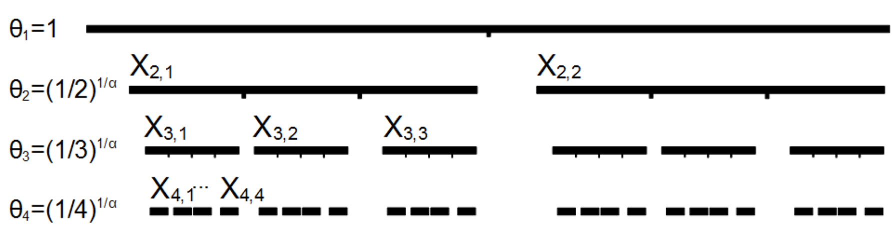
Here we put $c_\alpha = [1 - 2^{-(\frac{1}{\alpha}-1)}]/3$. Hence c_α gives a uniform lower bound for the portion of the gap that an interval of length θ_N can not fill in an interval of length N^{-1}.

For each N, pick $X_{N,j} \in [\frac{j-1}{N} + \frac{c_\alpha}{N}, \frac{j-1}{N} + \frac{2c_\alpha}{N}]$, $j = 1, \cdots, N$. Then we can correspondingly "dissect" $[0,1]$ into N disjoint intervals $[X_{N,j}, X_{N,j} + \theta_N]$.



Now start with $N = 2$, we "dissect" $[0,1]$ into two intervals $[X_{2,j_2}, X_{2,j_2} + \theta_2]$, $j_2 = 1, 2$. Then perform the dissection with $N = 3$ to each $[X_{2,j_2}, X_{2,j_2} + \theta_2]$, we get six intervals $[X_{2,j_2} + \theta_2 X_{3,j_3}, X_{2,j_2} + \theta_2 X_{3,j_3} + \theta_2\theta_3]$, $j_2 = 1, 2$, $j_3 = 1, 2, 3$. Continue the procedure, after the N-th step, we get $2 \cdot 3 \cdots N$ disjoint closed intervals of the form

$$[X_{2,j_2} + \cdots + \theta_2 \ldots \theta_{N-1} X_{N,j_N}, X_{2,j_2} + \cdots + \theta_2 \ldots \theta_{N-1} X_{N,j_N} + \theta_2 \ldots \theta_{N-1}\theta_N].$$



Denote by K_N the union of these intervals and set $K_X = \cap_N K_N$, where the index

$$X = (X_{N,j_N})_{\substack{N=2,3,\cdots \\ j_N=1,\cdots,N}}$$

Then K_X is a compact set.

1.4 The measure

Equip K_N with the uniform probability measure μ_N and let F_N be its distribution function. Since F_N is continuous and $\|F_N - F_{N+1}\|_\infty \le (N!)^{-1}$, F_N converges uniformly to a continuous distribution function F. Denote $\mu_X = dF$, the probability measure corresponding to F, then $\mu_X(K_X) = 1$.

1.5 The Fourier transform

Since μ_N converges weakly to μ_X, in particular, $\hat{\mu}_N(\xi) \to \hat{\mu}_X(\xi), \forall \xi$. Notice that for $\xi \neq 0$,

$$\begin{aligned}
\hat{\mu}_N(\xi) &= \frac{e^{i\xi\theta_2\cdots\theta_N} - 1}{i\xi\theta_2\cdots\theta_N} \frac{1}{N\cdots 2} \sum_{j_2,\cdots,j_N} e^{i\xi(X_{2,j_2}+\cdots+\theta_2\cdots\theta_{N-1}X_{N,j_N})} \\
&= \frac{e^{i\xi\theta_2\cdots\theta_N} - 1}{i\xi\theta_2\cdots\theta_N} \prod_{k=2}^{N} (\frac{1}{k}\sum_{j_k=1}^{k} e^{i\xi\theta_2\cdots\theta_{k-1}X_{k,j_k}})
\end{aligned}$$

Let $N \to \infty$ we get

$$\hat{\mu}_X(\xi) = \prod_{k=2}^{\infty} (\frac{1}{k}\sum_{j_k=1}^{k} e^{i\xi\theta_2\cdots\theta_{k-1}X_{k,j_k}}), \forall \xi.$$

1.6 Randomization

Now we randomize X such that $\{X_{N,j_N}, N = 2, 3, \cdots, j_N = 1, \cdots, N\}$ are independent and each X_{N,j_N} is uniformly distributed on $[\frac{j_N-1}{N} + \frac{c_\alpha}{N}, \frac{j_N-1}{N} + \frac{2c_\alpha}{N}]$. In what follows we suppress the subscript X.

1.7 From average decay to deterministic decay

It now suffices prove the Fourier decay estimate in the average sense. Precisely, for any $q, m \in \mathbb{N}, q, m \ge 1$, we will show that for some constant

$C = C(\alpha, m, q)$,
$$E[|\hat{\mu}(\xi)|^{2q}] \leq |\xi|^{-(1-\frac{1}{m})\alpha q}, \forall |\xi| \geq C.$$

Assuming this is proven, we can choose $q > 2m\alpha^{-1}$ and let $\xi = n \in \mathbb{Z}, |n| \geq C$ in the above inequality, then

$$E[|n|^{(1-\frac{2}{m})\alpha q}|\hat{\mu}(n)|^{2q}] \leq |n|^{-\frac{1}{m}\alpha q}$$

Summing over n, we get

$$E[\sum_{|n|\geq C} |n|^{(1-\frac{2}{m})\alpha q}|\hat{\mu}(n)|^{2q}] \leq \sum_{|n|\geq 1} |n|^{-2} < \infty$$

Hence
$$\hat{\mu}(n) = O(|n|^{-(1-\frac{2}{m})\frac{\alpha}{2}}), \text{ a.s.}$$

In order to pass from the integers to the reals, notice the following

Lemma 2 (cf. [3] p.252)**.** *Let μ be a probability measure supported on $[0, 1]$ and $\beta > 0$ such that $\hat{\mu}(n) = O(|n|^{-\beta})$, then $\hat{\mu}(\xi) = O(|\xi|^{-\beta})$.*

Applying this lemma we see that almost surely we have

$$\hat{\mu}(\xi) = O(|\xi|^{-(1-\frac{2}{m})\frac{\alpha}{2}}), \forall m.$$

1.8 The key estimate

To prove the average decay estimate we first estimate

$$\begin{aligned}
E[|\frac{1}{k}\sum_{j=1}^{k} e^{i\eta X_{k,j}}|^{2q}] &= \frac{1}{k^{2q}} E[(\sum_{j_1,\cdots,j_q=1}^{k} e^{i\eta(X_{k,j_1}+\cdots+X_{k,j_q})})(\sum_{i_1,\cdots,i_q=1}^{k} e^{-i\eta(X_{k,i_1}+\cdots+X_{k,i_q})})] \\
&= \frac{1}{k^{2q}} E[\sum_{j_1,\cdots,j_q=1}^{k} \sum_{\substack{\{i_1,\cdots,i_q\} \\ =\{j_1,\cdots,j_q\}}} 1] + \frac{1}{k^{2q}} E[\sum_{\substack{n_1,\cdots,n_k\in\mathbb{Z} \\ (n_1,\cdots,n_k)\neq 0}} e^{i\eta(n_1X_{k,1}+\cdots+n_kX_{k,k})}] \\
&\leq \frac{q!}{k^q} + \sup_{\substack{j=1,\cdots,k \\ n\in\mathbb{Z},n\neq 0}} |E[e^{i\eta n X_{k,j}}]| \\
&\leq \frac{q^q}{k^q} + 2c_\alpha^{-1} k|\eta|^{-1}
\end{aligned}$$

1.9 Proof of the average decay

Thus, if $2c_\alpha^{-1}k|\eta|^{-1} = 2c_\alpha^{-1}k|\xi\theta_2\cdots\theta_{k-1}|^{-1} \leq q^q k^{-q}$ for $k = 2, \cdots, N$, then

$$\begin{aligned} E[|\hat{\mu}(\xi)|^{2q}] &\leq E[\prod_{k=2}^{N} |\frac{1}{k}\sum_{j_k=1}^{k} e^{i\xi\theta_2\cdots\theta_{k-1}X_{k,j_k}}|^{2q}] \\ &= \prod_{k=2}^{N} E[|\frac{1}{k}\sum_{j_k=1}^{k} e^{i\xi\theta_2\cdots\theta_{k-1}X_{k,j_k}}|^{2q}] \\ &\leq \prod_{k=2}^{N} \frac{2q^q}{k^q} \leq \frac{2^N q^{qN}}{(N!)^q} = \left[\frac{(2^{\frac{1}{q}}q)^N}{N!}\right]^q \end{aligned}$$

The above condition holds if and only if it holds for N, or equivalently

$$2c_\alpha^{-1}q^{-q}N^{q+1}[(N-1)!]^{\frac{1}{\alpha}} \leq |\xi|$$

Let $N = N(\xi)$ be maximal such that the inequality is satisfied, then $N(\xi)$ is well defined for large $|\xi|$ and is increasing in $|\xi|$ with limit ∞ as $|\xi| \to \infty$. Moreover, due to maximality we have the opposite inequality for $N+1$. Raise each term to the α-th power we get

$$c_{\alpha,q}N^{\alpha q+\alpha}(N-1)! \leq |\xi|^\alpha \leq c_{\alpha,q}(N+1)^{\alpha q+\alpha}N!$$

where $c_{\alpha,q} = (2c_\alpha^{-1}q^{-q})^\alpha$. Hence,

$$\frac{(2^{\frac{1}{q}}q)^N}{N!} \leq c_{\alpha,q}(N+1)^{\alpha q+\alpha}|\xi|^{-\alpha}(2^{\frac{1}{q}}q)^N$$

Notice that for N large enough (depending on α, m, q) we have

$$(N+1)^{\alpha q+\alpha}, (2^{\frac{1}{q}}q)^N \leq [(N-1)!]^{\frac{1}{2m}}$$

Hence for $|\xi|$ large enough (depending on α, m, q),

$$
\begin{aligned}
E[|\hat{\mu}(\xi)|^{2q}]^{\frac{1}{q}} &\le \frac{(2^{\frac{1}{q}}q)^N}{N!} \\
&\le c_{\alpha,q}|\xi|^{-\alpha}(N+1)^{\alpha q+\alpha}(2^{\frac{1}{q}}q)^N \\
&\le c_{\alpha,q}|\xi|^{-\alpha}[(N-1)!]^{\frac{1}{m}} \\
&\le c_{\alpha,q}|\xi|^{-\alpha}c_{\alpha,q}^{-\frac{1}{m}}|\xi|^{\frac{\alpha}{m}} \\
&= c_{\alpha,m,q}|\xi|^{-(1-\frac{1}{m})\alpha}
\end{aligned}
$$

Since m can be arbitrarily large, one can rid of the constant by choosing larger $|\xi|$. Raise both sides to the power q, we get for some $C = C(\alpha, m, q)$

$$E[|\hat{\mu}(\xi)|^{2q}] \le |\xi|^{-(1-\frac{1}{m})\alpha q}, \forall |\xi| \ge C.$$

1.10 The dimensions

Let K be a compact set in $\mathbb{R}^1$, define the *Fourier dimension* of K by

$$\dim_F(K) = \sup\{\beta \in [0,1] : \exists \mu \in \mathbf{P}(K), \text{ s.t. } \hat{\mu}(\xi) = O(|\xi|^{-\beta/2})\}$$

where $\mathbf{P}(K)$ denotes the space of probability measures on K.

Lemma 3 (cf. [3] p.133)**.** *For any compact set K in $\mathbb{R}^1, \dim_F(K) \le \dim_H(K)$.*

Here $\dim_H(K)$ denotes the Hausdorff dimension of K. Finally, one can show that in the above construction, for any $K = K_X$ and $\mu = \mu_X$, $\dim_H(K) = \alpha$ and $\mu(I) \lesssim |I|^\alpha$ for all interval I.

References

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[2] Bluhm, C., *Random recursive construction of Salem sets.* Ark. Mat. **34** (1996), no. 1, 51–63;

[3] Kahane, J.-P., *Some Random Series of Functions.* 2nd ed, 1985, Cambridge University Press, Cambridge.

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