Salem-Bluhm's Construction of Salem Sets

Xianghong Chen (UW-Madison)

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Fourier Transform of Finite Measures

Define the Fourier transform of a probability measure μ on $\mathbb{R}^d$

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} d\mu(x), \forall \xi \in \mathbb{R}^d$$

When μ is absolutely continuous with respect to the Lebesgue measure,

$$\hat{\mu}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx$$

where f is an integrable function such that $\mu = f dx$.

Question: Behavior of $\hat{\mu}(\xi)$ as $|\xi| \to \infty$?

Fourier Decay: Absolutely Continuous Measures

Riemann-Lebesgue Lemma: If $f \in L^1$, then $\hat{f}(\xi) = o(1)$.

This follows from the fact that when $f = \chi_{[a,b]}$,

$$\widehat{\chi}_{[a,b]}(\xi) = \int_a^b e^{i\xi x} dx = \frac{e^{i\xi b} - e^{i\xi a}}{i\xi} = O(|\xi|^{-1})$$

Question: Rate of decay of $\hat{f}(\xi)$?

Fact: For any $0 < \alpha < 1$, there exists compact set K such that $\widehat{\chi}_K(\xi) \neq O(|\xi|^{-\alpha})$.

Fact: When f is continuous, the rate of decay of $\hat{f}(\xi)$ is closely related to the smoothness of f.

Fourier Decay: Singular Measures

What happens if μ is singular?

The simplest case is $\mu = \delta_a$, the Dirac measure at a,

$$\hat{\delta}_a(\xi) = \int e^{i\xi x} d\delta_a = e^{ia\xi}$$

which does not vanish at ∞. So the R-L lemma does not hold.

The next simplest case is $\mu = p_1 \delta_{a_1} + p_2 \delta_{a_2}$,

$$\hat{\mu}(\xi) = p_1 e^{ia_1\xi} + p_2 e^{ia_2\xi}$$

If a_1 and a_2 are commensurable, then $\hat{\mu}(\xi)$ is periodic, so again there is no decay.

Fourier Decay: Singular Measures

If a_1 and a_2 are incommensurable, say $a_1 = 1, a_2 = \sqrt{2}$, then

$$\hat{\mu}(2\pi k) = p_1 e^{i2\pi k} + p_2 e^{i\sqrt{2}\ 2\pi k} = p_1 + p_2 e^{i\sqrt{2}\ 2\pi k}$$

But $e^{i\sqrt{2}\ 2\pi k}$ can approximate any point on the circle as $k \to \infty$, so $\hat{\mu}(\xi)$ has no decay.

Question: Does there exist μ supported on a countable set, i.e. $\mu = \sum_k p_k \delta_{a_k}$, such that its Fourier transform

$$\hat{\mu}(\xi) = \sum_k p_k e^{ia_k \xi}$$

has decay at ∞?

Fourier Decay: Singular Measures

The answer is No. This can be seen from either of the following.

Wiener's Theorem:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\hat{\mu}(\xi)|^2 d\xi = \sum_x \mu(x)^2$$

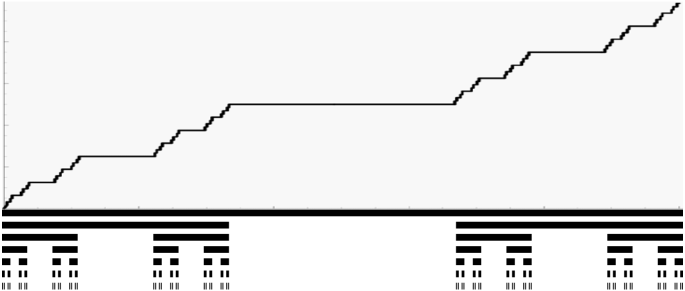
If $\hat{\mu}(\xi) = o(1)$, then LHS$= 0$. Conversely, if μ has no point mass, then its Fourier transform decays in the average sense.

Theorem: If a compact set K carries a measure μ with $\hat{\mu}(\xi) = O(|\xi|^{-\alpha})$ for some $\alpha > 0$, then there exists N such that the N-fold arithmetic sum $K + \cdots + K$ contains an interior point. In particular, K generates $\mathbb{R}$.

Fourier Decay: Cantor Set

Question: Does the converse of the above theorems hold? i.e. If the set A generates $\mathbb{R}$ and carries a measure μ that charges no point, dose $\hat{\mu}(\xi) = o(1)$ necessarily hold?

The answer is No. A counterexample is given by the standard $\frac{1}{3}$ Cantor set C. Notice that $C + C = [0, 2]$ and the Cantor function ψ is continuous.



Fourier Decay: Cantor Set

What is $\widehat{d\psi}(\xi)$?

One nice way to think of C and $\mu = d\psi$ is the so-called Bernoulli convolution. For convenience we translate the Cantor set so that it is centered at 0. Let $\{\epsilon_k, k \geq 1\}$ be an i.i.d. sequence with $P(\epsilon_k = -1) = P(\epsilon_k = 1) = \frac{1}{2}$. Set random variable

$$Y = \sum_{k=1}^{\infty} \frac{\epsilon_k}{3^k} = \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3^2} + \frac{\epsilon_3}{3^3} + \cdots$$

By the ternary expansion of points in Cantor set, the distribution of X coincides the Cantor measure μ. Hence

$$\mu = *_{k=1}^{\infty}\left(\frac{\delta_{-3^{-k}}}{2} + \frac{\delta_{3^{-k}}}{2}\right) = \lim_{n\to\infty} \sum_{\epsilon_1,\cdots,\epsilon_n=\pm 1} \frac{1}{2^n}\delta_{\frac{\epsilon_1}{3}+\cdots+\frac{\epsilon_n}{3^n}}$$

Fourier Decay: Cantor Set

$$\begin{aligned}\hat{\mu}(\xi) &= E[e^{i\xi X}] = E[e^{i\sum_k \xi \frac{\epsilon_k}{3^k}}] \\ &= \prod_k E[e^{i\xi \frac{\epsilon_k}{3^k}}] \\ &= \prod_k \frac{1}{2}(e^{i-\frac{\xi}{3^k}} + e^{i\frac{\xi}{3^k}}) \\ &= \prod_{k=1}^{\infty} \cos(\frac{\xi}{3^k})\end{aligned}$$

$$\hat{\mu}(3^n 2\pi) = [\prod_{k=1}^{n} \cos(\frac{3^n 2\pi}{3^k})][\prod_{k=1}^{\infty} \cos(\frac{2\pi}{3^k})] = \hat{\mu}(2\pi) \neq 0$$

So the Cantor measure has no Fourier decay.

Fourier Decay: θ-Cantor Set

Similarly, for $0 < \theta \leq \frac{1}{2}$, if we define random variable

$$Y_\theta = \sum_{k=1}^{\infty} \theta^k \epsilon_k$$

Then X_θ induces the uniform distribution μ_θ on the θ-Cantor set C_θ with dissection ratio θ and

$$\hat{\mu}_\theta(\xi) = \prod_{k=1}^{\infty} \cos(\theta^k \xi)$$

When $\theta = \frac{1}{2}$, $C_{\frac{1}{2}} = [-1, 1]$ and

$$\hat{\mu}_{\frac{1}{2}}(\xi) = \prod_{k=1}^{\infty} \cos(\frac{\xi}{2^k}) = \frac{\sin(\xi)}{\xi} = O(|\xi|^{-1})$$

Fourier Decay: θ-Cantor Set

When $\theta < \frac{1}{2}$, it turns out that the asymptotic behavior of $\hat{\mu}_\theta(\xi)$ is determined by the number-theoretic properties of θ.

Erdős-Salem-Zygmund Theorem: TFAE,
(i) $\hat{\mu}_\theta(\xi) \neq o(1)$
(ii) For any μ supported on C_θ, $\hat{\mu}(\xi) \neq o(1)$
(iii) θ^{-1} is a PV number
(iv) C_θ is a set of uniqueness

A (Pisot-Vijayaraghavan) PV number is an algebraic number great than 1 whose conjugates are all inside the unit disc.

A set E is called a set of uniqueness if a Fourier series converges to 0 outside E, then all the Fourier coefficients are 0.

Fourier Decay: θ-Cantor Set

To see how the number-theoretic condition (iii) comes in, notice that by the same argument, for any $\lambda \neq 0$,

$$\lim_{n\to\infty} \hat{\mu}_\theta(\lambda\theta^{-n}2\pi) = [\prod_{k=1}^{\infty} \cos(\lambda\theta^{-k}2\pi)]\hat{\mu}_\theta(\lambda 2\pi)$$

Denote by $\|t\|$ the distance from t to its nearest integer, then

$$\prod_{k=1}^{\infty} \cos(\lambda\theta^{-k}2\pi) = \prod_{k=1}^{\infty} \cos(\|\lambda\theta^{-k}\|2\pi) \sim \prod_{k=1}^{\infty}(1 - 2\sin^2(\|\lambda\theta^{-k}\|\pi))$$
$$\sim \sum_{k=1}^{\infty} \|\lambda\theta^{-k}\|^2$$

Convergence of the last series characterizes θ^{-1} as a PV number.

Salem Sets: Definition

Question: Rate of decay of $\hat{\mu}_\theta(\xi)$?

Theorem (Frostman): If $K \subset \mathbb{R}^d$ is an α-dimensional compact set with $\alpha < d$, and μ is supported on K such that $|\hat{\mu}(\xi)|^2 = O(|\xi|^{-\beta})$, then $\beta \leq \alpha$.

Here the square can be understood via Plancherel theorem. If $\beta > d$, then $\hat{\mu} \in L^2(\mathbb{R}^d)$ which contradicts the fact that μ is singular.

The Fourier dimension of K is defined by

$$\dim_F(K) = \sup\{\beta \leq d : \exists \mu \text{ s.t. } |\hat{\mu}(\xi)|^2 = O(|\xi|^{-\beta})\}$$

By the above theorem $\dim_F(K) \leq \dim_H(K)$. When the equality holds, K is called a Salem set.

Examples: Non-Salem Sets

Example of non-Salem set in $\mathbb{R}^2$: Let $K = [0, 1] \subset \mathbb{R}^2$, then for any μ supported on K,

$$\hat{\mu}(\xi_1, \xi_2) = \int_{[0,1]} e^{i\xi_1 x_1 + \xi_2 0} d\mu(x_1) = \hat{\mu}(\xi_1)$$

So $\dim_F(K) = 0 < \dim_H(K) = 1$. (This can also be seen from the fact that K generates $\mathbb{R}$).

Example of non-Salem set in $\mathbb{R}^1$: If θ^{-1} is a PV number, then $\dim_F(C_\theta) = 0 < \dim_H(C_\theta) = \log_\theta(1/2)$.

Examples: Salem Sets

0-dimensional Salem sets: All the 0-dimensional sets are Salem sets since $\dim_F(K) \leq \dim_H(K)$.

d-dimensional Salem sets: Any open set in $\mathbb{R}^d$ is a Salem set since we can always choose a bump function supported in it. Is every set of positive measure a Salem set?

(d-1)-dimensional Salem sets: Let $K = S^{d-1} \subset \mathbb{R}^d$, then $\hat{\mu}(\xi) = O(|\xi|^{-\frac{d-1}{2}})$. So $\dim_F(K) = \dim_H(K) = d - 1$. Note that this rate of decay can never be improved by choosing different measure. (Note also that $S^{d-1} + S^{d-1} = B(0, 2)$.)

Examples: θ-Cantor Set

Return to the question. We have known that when θ^{-1} is not a PV number, $\hat{\mu}_\theta(\xi) = o(1)$. What is its rate of decay?

Fact: There exists θ (can be arbitrarily small) such that $\hat{\mu}_\theta(\xi) = o(1)$ but $\hat{\mu}_\theta(\xi) \neq O(|\xi|^{-\beta})$ for any $\beta > 0$. However, such θ constitute a set of fractional dimension.

Question: Does there exist θ such that such that C_θ is a Salem set?

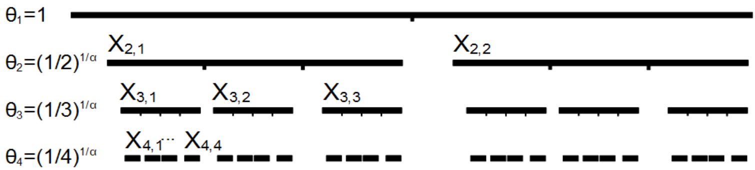
No.

Salem's Construction

Question (Beurling asked Salem): Given $\alpha \in (0, 1)$, does there exists Salem set of dimension α?

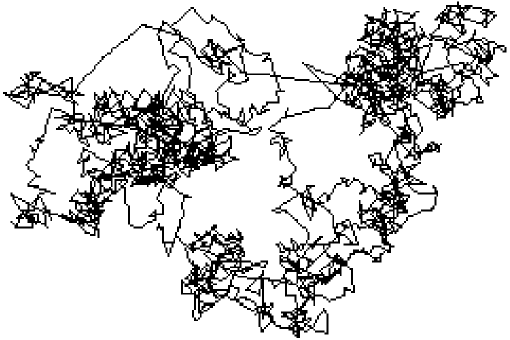
Salem's construction: Cantor set with randomized dissection ratios, (deterministic) incommensurable translations, and increasing dissection numbers.

Bluhm's variant: Cantor set with randomized translations and increasing dissection numbers.



Kahane's Salem Set

Kahane's Salem set: If $2\alpha < d$ and $K \subset \mathbb{R}_+$ is of dimension α, then the image of K under the d-dimensional Brownian motion is a.s. a 2α-dimensional Salem set.



Kaufman's Salem Set

The first deterministic Salem set was found by Kaufman.

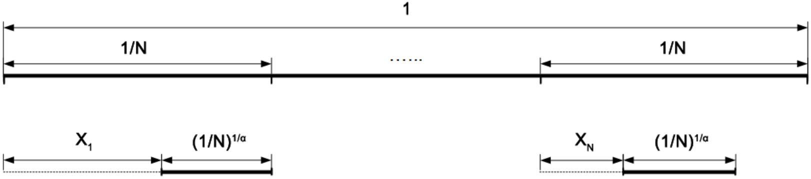
Kaufman's Salem set: For $\alpha > 0$, let $E(\alpha)$ be the set of real numbers x such that $\|nx\| \leq n^{-1-\alpha}$ for infinitely many n, where $\|t\|$ denotes the distance from t to the nearest integer. Then $E(\alpha)$ is a Salem set of dimension $\frac{2}{2+\alpha}$.

Note that the above condition is the same as

$$|x - \frac{a}{q}| \leq q^{-(2+\alpha)}$$

for infinitely many rationals $\frac{a}{q}$.

Salem-Bluhm's Construction



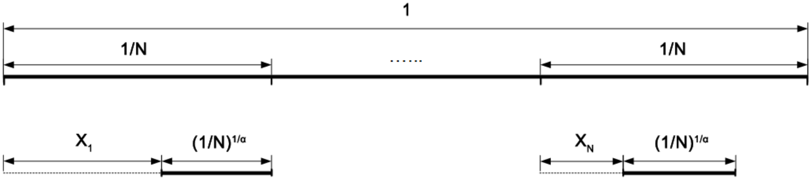
$$\frac{1}{N} - \frac{1}{N^{1/\alpha}} = (1 - N^{-\frac{1}{\alpha}+1})\frac{1}{N}$$
$$\geq (1 - 2^{-\frac{1}{\alpha}+1})\frac{1}{N}$$
$$> \frac{c_\alpha}{N}$$

Where $0 < c_\alpha < 1 - 2^{-\frac{1}{\alpha}+1}$ is fixed. As long as

$$0 \leq X_j \leq \frac{c_\alpha}{N}, j = 1, \cdots, N$$

we get N disjoint intervals.

Salem-Bluhm's Construction



For each N, fix $(X_{N,1}, \cdots, X_{N,N})$ with

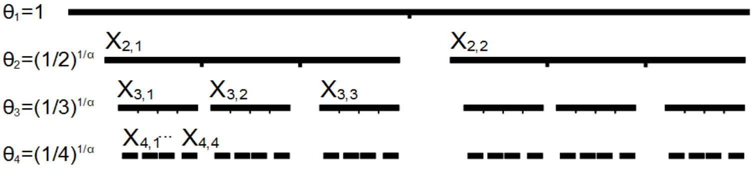
$$0 \leq X_{N,j} \leq \frac{c_\alpha}{N}, j = 1, \cdots, N$$

Start with interval $[0, 1]$ and $N = 2$, we get 2 intervals

$$[X_{2,1}, X_{2,1} + 2^{-1/\alpha}], \frac{1}{2} + [X_{2,2}, X_{2,2} + 2^{-1/\alpha}]$$

Then apply to each of these two intervals with $N = 3$, we get 6 intervals of size $2^{-1/\alpha}3^{-1/\alpha}$... At the step N, we get $N!$ intervals of size $(N!)^{-1/\alpha}$.

Cantor Set C_X



Denote by K_N the union of these intervals and set $C_X = \cap_N K_N$, where

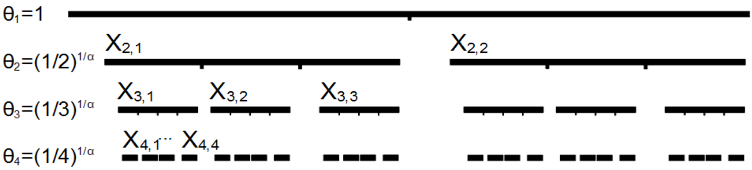
$$X = (X_{N,j})_{\substack{N=2,3,\cdots \\ j=1,\cdots,N}}$$

Since K_N is a covering of C_X by $N!$ intervals of size $(N!)^{-1/\alpha}$

$$H_\alpha(C_X) \leq \lim_{N\to\infty} N![(N!)^{-1/\alpha}]^\alpha = 1$$

So $\dim_H(C_X) \leq \alpha$.

Cantor Measure μ_X

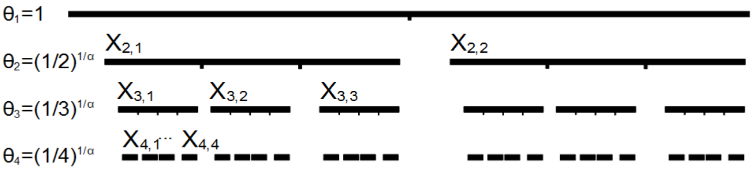


For notational convenience we set $\theta_N = N^{-1/\alpha}$ and reset $X_{N,j} = X_{N,j} + \frac{j-1}{N}$. The convolution

$$\mu_X = *_{k=2}^{\infty}\left(\sum_{j=1}^{k} \frac{1}{k}\delta_{\theta_1 \cdots \theta_{k-1} X_{k,j}}\right)$$

gives the uniform (probability) measure on C_X.

Fourier Transform of μ_X



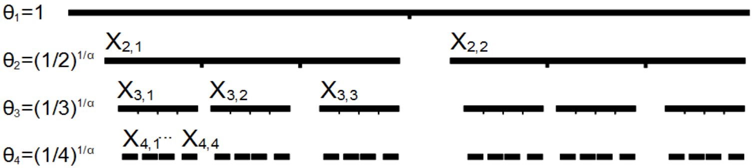
$$\hat{\mu}_X(\xi) = \prod_{k=2}^{\infty} \mathcal{F}\Big(\sum_{j=1}^{k} \frac{1}{k} \delta_{\theta_1 \cdots \theta_{k-1} X_{k,j}}\Big)(\xi)$$

$$= \prod_{k=2}^{\infty} \Big(\frac{1}{k} \sum_{j=1}^{k} e^{i\xi\theta_1 \cdots \theta_{k-1} X_{k,j}}\Big)$$

We want

$$\hat{\mu}_X(\xi) = O(|\xi|^{-(1-\frac{1}{m})\frac{\alpha}{2}}), \forall m$$

Randomizing X



Now randomize $X_{N,j}$ such that

$$X_{N,j} \sim \left(\frac{c_\alpha}{N}\right)^{-1} \chi_{\left[\frac{j-1}{N}, \frac{j-1}{N} + \frac{c_\alpha}{N}\right]}$$

Further, make $X_{N,j}, N = 2, 3, \cdots; j = 1, \cdots, N$ independent.

We end up with a random Cantor set C_X together with its uniform distribution μ_X. We will suppress the subscript X.

From Average to Deterministic Decay

We first prove the desired Fourier decay estimate in the average sense. Precisely, for any $q, m \in \mathbb{N}_{\geq 1}$, we will show that for some constant $C = C(\alpha, m, q)$,

$$E[|\hat{\mu}(\xi)|^{2q}] \leq |\xi|^{-(1-\frac{1}{m})\alpha q}, \forall |\xi| \geq C.$$

If this is proved, choose $q > 2m\alpha^{-1}$ and let $\xi = n \in \mathbb{Z}, |n| \geq C$, we get

$$E[|n|^{(1-\frac{2}{m})\alpha q}|\hat{\mu}(n)|^{2q}] \leq |n|^{-\frac{1}{m}\alpha q}$$

$$E[\sum_{|n|\geq C} |n|^{(1-\frac{2}{m})\alpha q}|\hat{\mu}(n)|^{2q}] \leq \sum_{|n|\geq 1} |n|^{-2} < \infty$$

$$\hat{\mu}(n) = O(|n|^{-(1-\frac{2}{m})\frac{\alpha}{2}}), \text{ a.s.}$$

From n to ξ

Lemma: Let μ be a probability measure supported on $[0,1]$ and $\beta > 0$ such that $\hat{\mu}(n) = O(|n|^{-\beta})$, then $\hat{\mu}(\xi) = O(|\xi|^{-\beta})$.

Now since

$$\hat{\mu}(n) = O(|n|^{-(1-\frac{2}{m})\frac{\alpha}{2}}), \text{ a.s.}$$

We obtain

$$\hat{\mu}(\xi) = O(|\xi|^{-(1-\frac{2}{m})\frac{\alpha}{2}}), \forall m, \text{ a.s.}$$

Hence $\alpha \leq \dim_F(C), \text{ a.s.}$

Combine this with $\dim_F(C) \leq \dim_H(C) \leq \alpha$, we see that C is almost surely an α-dimensional Salem set.

Proof of The Average Decay

For any $N \geq 2$,

$$|\hat{\mu}(\xi)| = \prod_{k=2}^{\infty} |\frac{1}{k}\sum_{j=1}^{k} e^{i\xi\theta_1\cdots\theta_{k-1}X_{k,j}}|$$

$$\leq \prod_{k=2}^{N} |\frac{1}{k}\sum_{j=1}^{k} e^{i\xi\theta_1\cdots\theta_{k-1}X_{k,j}}|$$

$$|\hat{\mu}(\xi)|^{2q} \leq \prod_{k=2}^{N} |\frac{1}{k}\sum_{j=1}^{k} e^{i\xi\theta_1\cdots\theta_{k-1}X_{k,j}}|^{2q}$$

$$E[|\hat{\mu}(\xi)|^{2q}] \leq \prod_{k=2}^{N} E[|\frac{1}{k}\sum_{j=1}^{k} e^{i\xi\theta_1\cdots\theta_{k-1}X_{k,j}}|^{2q}]$$

The Key Estimate

$$
\begin{aligned}
& E[|\frac{1}{k}\sum_{j=1}^{k} e^{i\eta X_{k,j}}|^{2q}] \\
&= \frac{1}{k^{2q}} E[(\sum_{j_1,\cdots,j_q=1}^{k} e^{i\eta(X_{k,j_1}+\cdots+X_{k,j_q})})(\sum_{i_1,\cdots,i_q=1}^{k} e^{-i\eta(X_{k,i_1}+\cdots+X_{k,i_q})})] \\
&= \frac{1}{k^{2q}} E[\sum_{j_1,\cdots,j_q=1}^{k} \sum_{\substack{\{i_1,\cdots,i_q\} \\ =\{j_1,\cdots,j_q\}}} 1] + \frac{1}{k^{2q}} E[\sum_{\substack{n_1,\cdots,n_k\in\mathbb{Z} \\ (n_1,\cdots,n_k)\neq 0}} e^{i\eta(n_1 X_{k,1}+\cdots+n_k X_{k,k})}] \\
&\leq \frac{q!}{k^q} + \sup_{\substack{j=1,\cdots,k \\ n\in\mathbb{Z}, n\neq 0}} |E[e^{i\eta n X_{k,j}}]| \\
&\leq \frac{q^q}{k^q} + 2c_\alpha^{-1} k|\eta|^{-1}
\end{aligned}
$$

Truncation

If for $k = 2, \cdots, N$,

$$2c_\alpha^{-1} k |\xi\theta_1 \ldots \theta_{k-1}|^{-1} \le \frac{q^q}{k^q}$$

Then

$$\begin{aligned} E[|\hat{\mu}(\xi)|^{2q}] &\le \prod_{k=2}^{N} E[|\frac{1}{k}\sum_{j=1}^{k} e^{i\xi\theta_1\cdots\theta_{k-1}X_{k,j}}|^{2q}] \\ &\le \prod_{k=2}^{N}(\frac{q^q}{k^q} + 2c_\alpha^{-1} k |\xi\theta_1 \ldots \theta_{k-1}|^{-1}) \\ &\le \prod_{k=2}^{N} \frac{2q^q}{k^q} = \frac{2^N q^{qN}}{(N!)^q} = \left[\frac{(2^{\frac{1}{q}} q)^N}{N!}\right]^q \end{aligned}$$

$N! \sim |\xi|^\alpha$

Notice that

$$2c_\alpha^{-1} k|\xi\theta_1 \dots \theta_{k-1}|^{-1} \le \frac{q^q}{k^q}$$

is equivalent to

$$2c_\alpha^{-1} q^{-q} k^{q+1} [(k-1)!]^{\frac{1}{\alpha}} \le |\xi|$$

and the LHS is increasing in k. So the inequality holds for $k = 2, \cdots, N$ if and only if it holds for N, i.e.

$$2c_\alpha^{-1} q^{-q} N^{q+1} [(N-1)!]^{\frac{1}{\alpha}} \le |\xi|$$

Let $N = N(\xi)$ be the maximal one, we get,

$$c_{\alpha,q} N^{\alpha q+\alpha} (N-1)! \le |\xi|^\alpha \le c_{\alpha,q} (N+1)^{\alpha q+\alpha} N!$$

End of The Proof

From $c_{\alpha,q}N^{\alpha q+\alpha}(N-1)! \leq |\xi|^\alpha \leq c_{\alpha,q}(N+1)^{\alpha q+\alpha}N!$, we get

$$(N-1)! \leq c_{\alpha,q}^{-1}|\xi|^\alpha, \quad \frac{1}{N!} \leq c_{\alpha,q}|\xi|^{-\alpha}(N+1)^{\alpha q+\alpha}$$

Notice that for N large enough (depending on α, m, q),

$$(N+1)^{\alpha q+\alpha}, \quad (2^{\frac{1}{q}}q)^N \leq [(N-1)!]^{\frac{1}{2m}}$$

Hence for ξ large enough (depending on α, m, q), we have

$$\begin{aligned} E[|\hat{\mu}(\xi)|^{2q}]^{\frac{1}{q}} &\leq \frac{(2^{\frac{1}{q}}q)^N}{N!} \\ &\leq c_{\alpha,q}|\xi|^{-\alpha}(N+1)^{\alpha q+\alpha}(2^{\frac{1}{q}}q)^N \\ &\leq c_{\alpha,q}|\xi|^{-\alpha}[(N-1)!]^{\frac{1}{m}} \\ &\leq c_{\alpha,q}|\xi|^{-\alpha}c_{\alpha,q}^{-\frac{1}{m}}|\xi|^{\frac{\alpha}{m}} = c_{\alpha,m,q}|\xi|^{-(1-\frac{1}{m})\alpha} \end{aligned}$$