Let $n \geq 1$ be a fixed integer, and let

$$S_j = \{(\xi, \frac{1}{2}|\xi|^2) \in \mathbb{R}^n \times \mathbb{R} : |\xi - (-1)^j e_1| < 1/2\}$$

where $j = 1, 2$ and $e_1 = (1, 0, \cdots, 0)$. Denote by σ_j the surface measure on S_j. Our goal is to prove the following.

Theorem 1 (L^2-bilinear restriction theorem)**.**

$$\|\widehat{f_1\sigma_1}\widehat{f_2\sigma_2}\|_{L^q(\mathbb{R}^{n+1})} \leq C_{n,q}\|f_1\|_{L^2(\sigma_1)}\|f_2\|_{L^2(\sigma_2)} \tag{1}$$

for any $q > \frac{n+3}{n+1}$.

This range of q is sharp up to endpoint.

Lemma 2 (epsilon-removal)**.** *To prove Theorem 1, it suffices to prove*

$$\|\widehat{f_1\sigma_1}\widehat{f_2\sigma_2}\|_{L^{\frac{n+3}{n+1}}(B(0,R))} \leq C_{n,\alpha}R^\alpha\|f_1\|_{L^2(\sigma_1)}\|f_2\|_{L^2(\sigma_2)}, \ R \geq 1 \tag{2}$$

for all $\alpha > 0$.

Denote (2) by

$$R^*(2 \times 2 \to \frac{n+3}{n+1}, \alpha).$$

Lemma 2 will follow from iterating the following.

Lemma 3 (induction on scale)**.**

$$R^*\big(2 \times 2 \to \frac{n+3}{n+1}, \alpha\big) \Rightarrow R^*\big(2 \times 2 \to \frac{n+3}{n+1}, \max((1-\delta)\alpha, C\delta)\big) \tag{3}$$

for any $\delta > 0$*, where* C *is independent of* δ*.*

The main tool for proving Lemma 3 is the following.

Lemma 4 (wave packet decomposition)**.**

$$\widehat{f_j\sigma_j} = \sum_{T_j} c_{T_j}\phi_{T_j}$$

where (i) T_j *ranges over* $\sqrt{R} \times \cdots \times \sqrt{R} \times \infty$ *tubes with initial positions*

$$x(T_j) \in R^{1/2}\mathbb{Z}^n \times \{0\}$$

and pointing in directions

$$\xi(T_j) \in R^{-1/2}\mathbb{Z}^n \times \{-1\}$$

that are normal to S_j*, i.e.*

$$(\xi(T_j), \frac{1}{2}|\xi(T_j)|^2) \in S_j.$$

(ii) The wave packet ϕ_{T_j} *is essentially supported on* T_j*, and* $\hat{\phi}_{T_j}$ *is supported in an* $O(R^{-1/2})$*-neighborhood of* $(\xi(T_j), \frac{1}{2}|\xi(T_j)|^2)$ *in* S_j*.*

(iii) The coefficients c_{T_j} *satisfy Bessel's inequality*

$$\Big(\sum_{T_j} |c_{T_j}|^2\Big)^{1/2} \approx \|f_j\|_2.$$

Since $\phi_{T_j}(x,t)$ solves the free Schrödinger equation, by the Plancherel theorem we have the following.

Lemma 5 (conservation of energy)**.**

$$\| \sum_{T_j \in \mathbf{T}_j} \phi_{T_j}(\cdot, t)\|_{L^2(\mathbb{R}^n)} = \| \sum_{T_j \in \mathbf{T}_j} \phi_{T_j}(\cdot, 0)\|_{L^2(\mathbb{R}^n)} \approx (\#\mathbf{T}_j)^{1/2} \tag{4}$$

for all $t \in \mathbb{R}$.

Now assuming $R^*\big(2 \times 2 \to \frac{n+3}{n+1}, \alpha\big)$ and $\|f_j\|_2 = 1$, we need to show

$$\| \sum_{T_1} \sum_{T_2} c_{T_1} c_{T_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n+3}{n+1}}(B(0,R))} \lessapprox R^{(1-\delta)\alpha} + R^{C\delta}.$$

By a pigeonholing argument, we may assume that $c_{T_1} = c_1$, $c_{T_2} = c_2$.

Lemma 6 (pigeonhole principle)**.** *It suffices to show*

$$\| \sum_{T_1 \in \mathbf{T}_1} \sum_{T_2 \in \mathbf{T}_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n+3}{n+1}}(B_R)} \lessapprox (R^{(1-\delta)\alpha} + R^{C\delta})(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2} \tag{5}$$

for any collections $\mathbf{T}_j$ *of* T_j, $j = 1, 2$.

To utilize the induction hypothesis, we split B_R into $O(R^{C\delta})$ many balls B of radius $R^{1-\delta}$. Denote by $\mathcal{B}$ the collection of such balls. We can thus estimate the left-hand side of (5) by

$$\sum_{B \in \mathcal{B}} \| \sum_{T_1 \in \mathbf{T}_1} \sum_{T_2 \in \mathbf{T}_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n+3}{n+1}}(B)}.$$

For each $T_1 \in \mathbf{T}_1$, we associate $\lessapprox 1$ many balls $B \in \mathcal{B}$ which "captures most intersection" between T_1 and $\mathbf{T}_2$; write $T_1 \sim B$ (and similarly for $T_2 \in \mathbf{T}_2$). We will prove (5) by showing the following estimates.

Lemma 7 (local part)**.**

$$\sum_{B \in \mathcal{B}} \| \sum_{\substack{T_1 \in \mathbf{T}_1 \\ T_1 \sim B}} \sum_{\substack{T_2 \in \mathbf{T}_2 \\ T_2 \sim B}} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n+3}{n+1}}(B)} \lessapprox R^{(1-\delta)\alpha} (\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}. \tag{6}$$

Lemma 8 (global part)**.**

$$\| \sum_{\substack{T_1 \in \mathbf{T}_1 \\ T_1 \nsim B}} \sum_{T_2 \in \mathbf{T}_2} \phi_{T_1} \phi_{T_2} \|_{L^{\frac{n+3}{n+1}}(B)} \lessapprox R^{C\delta} (\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}. \tag{7}$$

Proof of Lemma 7. Applying the induction hypothesis to each ball B, we can estimate the left-hand side of (6) by

$$\sum_{B\in\mathcal{B}} R^{(1-\delta)\alpha}\big(\#\{T_1\in\mathbf{T}_1 : T_1\sim B\}\big)^{1/2}\big(\#\{T_2\in\mathbf{T}_2 : T_2\sim B\}\big)^{1/2}.$$

By the Cauchy-Schwarz inequality, this can be bounded by

$$R^{(1-\delta)\alpha}\Big(\sum_{B\in\mathcal{B}}\sum_{T_1\in\mathbf{T}_1} 1_{T_1\sim B}\Big)^{1/2}\Big(\sum_{B\in\mathcal{B}}\sum_{T_2\in\mathbf{T}_2} 1_{T_2\sim B}\Big)^{1/2}.$$

which by Fubini's theorem becomes

$$\begin{aligned} &R^{(1-\delta)\alpha}\Big(\sum_{T_1\in\mathbf{T}_1}\sum_{B\in\mathcal{B}} 1_{T_1\sim B}\Big)^{1/2}\Big(\sum_{T_2\in\mathbf{T}_2}\sum_{B\in\mathcal{B}} 1_{T_2\sim B}\Big)^{1/2}\\ &\lessapprox R^{(1-\delta)\alpha}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}, \end{aligned}$$

as desired. □

Lemma 8 follows from interpolating between the following estimates.

Lemma 9 (L^1-estimate)**.**

$$\|\sum_{\substack{T_1\in\mathbf{T}_1\\ T_1\nsim B}}\sum_{T_2\in\mathbf{T}_2}\phi_{T_1}\phi_{T_2}\|_{L^1(B)}\lesssim R(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}.$$

Proof. By Hölder's inequality it suffices to show

$$\|\sum_{T_j\in\mathbf{T}_j}\phi_{T_j}\|_{L^2(B)}\lesssim R^{1/2}(\#\mathbf{T}_j)^{1/2}$$

where $j = 1, 2$. But this follows directly from (4) and integrating in t. □

Lemma 10 (L^2-estimate)**.**

$$\|\sum_{\substack{T_1\in\mathbf{T}_1\\ T_1\nsim B}}\sum_{T_2\in\mathbf{T}_2}\phi_{T_1}\phi_{T_2}\|_{L^2(B)}\lessapprox R^{C\delta}R^{-(n-1)/4}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}. \tag{8}$$

To prove Lemma 10, we start with the following observation.

Lemma 11 (transversal waves)**.**

$$\|\phi_{T_1}\phi_{T_2}\|_{L^2(\mathbb{R}^{n+1})}\lesssim R^{-(n-1)/4}. \tag{9}$$

Proof. Use Plancherel's theorem as in Wolff's paper. □

We now split B_R into balls of radius $\sqrt{R}$. Denote by $\mathbf{q}$ the collection of such balls, and denote

$$\mathbf{q}(B) := \{q\in\mathbf{q} : q\subset B\}.$$

For $q\in\mathbf{q}$, denote

$$\mathbf{T}_j(q) := \{T_j\in\mathbf{T}_j : T_j\cap R^{\delta}q\neq\emptyset\},\ j=1,2$$
$$\mathbf{T}_1^{\nsim B}(q) := \{T_j\in\mathbf{T}_1(q) : T_1\nsim B\}.$$

By a pigeonholing argument, we may assume that

$$\#\mathbf{T}_j(q) \approx \mu_j, \ j = 1, 2$$
$$\#\{q \in \mathbf{q} : R^\delta q \cap T_1\} \approx \lambda_1.$$

Lemma 10 will follow from the following.

Lemma 12 (fine-scale decomposition)**.**

$$\sum_{q \in \mathbf{q}(B)} \| \sum_{T_1 \in \mathbf{T}_1^{\not\sim B}(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1}\phi_{T_2}\|^2_{L^2(q)} \lessapprox R^{C\delta} R^{-(n-1)/2} (\#\mathbf{T}_1)(\#\mathbf{T}_2). \tag{10}$$

To estimate the left-hand side, we expand

$$\| \sum_{T_1 \in \mathbf{T}_1^{\not\sim B}(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1}\phi_{T_2}\|^2_{L^2(\mathbb{R}^{n+1})}$$

as

$$\sum_{T_1 \in \mathbf{T}_1^{\not\sim B}(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \sum_{T_1' \in \mathbf{T}_1^{\not\sim B}(q)} \sum_{T_2' \in \mathbf{T}_2(q)} \int_{\mathbb{R}^{n+1}} \phi_{T_1}\phi_{T_2}\overline{\phi_{T_1'}\phi_{T_2'}} dxdt. \tag{11}$$

Since the Fourier transform of the integrand is supported near

$$(\xi_1, \frac{1}{2}|\xi_1|^2) + (\xi_2, \frac{1}{2}|\xi_2|^2) - (\xi_1', \frac{1}{2}|\xi_1'|^2) - (\xi_2', \frac{1}{2}|\xi_2'|^2),$$

we see that the integral vanishes unless

$$\xi_1 + \xi_2 = \xi_1' + \xi_2'$$
$$|\xi_1|^2 + |\xi_2|^2 = |\xi_1'|^2 + |\xi_2'|^2.$$

Lemma 13 (geometric constraint)**.** *The last two equations imply*

$$(\xi_1' - \xi_1) = (\xi_2 - \xi_2') \perp (\xi_1' - \xi_2) = (\xi_1 - \xi_2').$$

For fixed ξ_1 and ξ_2', denote

$$\pi(\xi_1, \xi_2') := \{\xi_1' : (\xi_1' - \xi_1) \perp (\xi_1 - \xi_2')\}$$

and

$$\nu(q) := \max_{\xi_1, \xi_2'} \#\{T_1' \in \mathbf{T}_1^{\not\sim B}(q) : \xi_1' \in \pi(\xi_1, \xi_2')\}.$$

Lemma 14 (fine-scale estimate)**.**

$$\| \sum_{T_1 \in \mathbf{T}_1^{\not\sim B}(q)} \sum_{T_2 \in \mathbf{T}_2(q)} \phi_{T_1}\phi_{T_2}\|^2_{L^2(q)} \lessapprox R^{C\delta} R^{-(n-1)/2} \nu(q) (\#\mathbf{T}_1^{\not\sim B}(q))(\#\mathbf{T}_2(q)).$$

Proof. This now follows from (11) and (9). □

To conclude the proof of Theorem 1, it remains to prove the following (Kakeya-type) combinatorial estimate.

Lemma 15 (coarse-scale estimate)**.**

$$\sum_{q \in \mathbf{q}(B)} \nu(q) (\#\mathbf{T}_1(q))(\#\mathbf{T}_2(q)) \lessapprox R^{C\delta} (\#\mathbf{T}_1)(\#\mathbf{T}_2).$$

Notice that by our assumption, $\#\mathbf{T}_2(q) \approx \mu_2$ and

$$\begin{aligned}\sum_{q \in \mathbf{q}(B)} \#\mathbf{T}_1(q) &\leq \sum_{q \in \mathbf{q}(B)} \sum_{T_1 \in \mathbf{T}_1} 1_{T_1 \cap R^\delta q \neq \emptyset} \\ &\approx \sum_{T_1 \in \mathbf{T}_1} \lambda_1 \\ &= (\#\mathbf{T}_1)\lambda_1.\end{aligned}$$

So, to prove Lemma 15, it suffices to show the following.

Lemma 16.

$$\nu(q_0) \lessapprox R^{C\delta} \frac{\#\mathbf{T}_2}{\lambda_1 \mu_2}$$

for all $q_0 \in \mathbf{q}(B)$.

Proof. We will use a "bush" argument centered at q_0. Fix ξ_1 and ξ_2'. Denote

$$\mathbf{T}_1' := \{T_1' \in \mathbf{T}_1^{\sim B}(q_0) : \xi_1' \in \pi(\xi_1, \xi_2')\}.$$

We need to show that

$$\#\mathbf{T}_1' \lessapprox R^{C\delta} \frac{\#\mathbf{T}_2}{\lambda_1 \mu_2}.$$

Consider the incidence set

$$\begin{aligned}\mathbf{I} =& \big\{(q, T_1', T_2) \in \mathbf{q} \times \mathbf{T}_1' \times \mathbf{T}_2 : \\ & R^\delta q \cap T_1' \neq \emptyset, R^\delta q \cap T_2 \neq \emptyset, \operatorname{dist}(q, q_0) \gtrapprox R^{-C\delta} R\big\}.\end{aligned}$$

On the one hand, we have

$$\begin{aligned}\#\mathbf{I} &= \sum_{T_1' \in \mathbf{T}_1'} \sum_{\substack{q \in \mathbf{q}: R^\delta q \cap T_1' \neq \emptyset \\ \operatorname{dist}(q, q_0) \gtrapprox R^{-C\delta} R}} \sum_{\substack{T_2 \in \mathbf{T}_2 \\ R^\delta q \cap T_2 \neq \emptyset}} 1 \\ &\approx \sum_{T_1' \in \mathbf{T}_1'} \sum_{\substack{q \in \mathbf{q}: R^\delta q \cap T_1' \neq \emptyset \\ \operatorname{dist}(q, q_0) \gtrapprox R^{-C\delta} R}} \mu_2 \\ &\gtrapprox \sum_{T_1' \in \mathbf{T}_1'} R^{-C\delta} \lambda_1 \mu_2 \\ &= (\#\mathbf{T}_1') R^{-C\delta} \lambda_1 \mu_2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\#\mathbf{I} &= \sum_{T_2 \in \mathbf{T}_2} \sum_{T_1' \in \mathbf{T}_1'} \sum_{\substack{q \in \mathbf{q}: R^\delta q \cap T_1' \neq \emptyset, R^\delta q \cap T_2 \neq \emptyset \\ \operatorname{dist}(q, q_0) \gtrapprox R^{-C\delta} R}} 1 \\ &\lessapprox \sum_{T_2 \in \mathbf{T}_2} \sum_{\substack{T_1' \in \mathbf{T}_1' \\ \exists q \in \mathbf{q}: R^\delta q \cap T_1' \neq \emptyset, R^\delta q \cap T_2 \neq \emptyset, \operatorname{dist}(q, q_0) \gtrapprox R^{-C\delta} R}} R^{C\delta} \\ &\lessapprox (\#\mathbf{T}_2) R^{C\delta}.\end{aligned}$$

Combining these two estimates for $\#\mathbf{I}$ gives the desired result. □